Problem Chosen	2022 MCM/ICM Summary Sheet	Team Control Number
A		2215067

Winning a Road Cycling Time Trial:
Optimal Strategy from Power Distribution Models

Abstract

Time Trial is one of road cycling races, including individual time trial (ITT) and team time trial (TTT). In this paper, we focus on power distribution during competition, establish mathematical models as well as simulate, and give the optimal strategy for riders in ITT and TTT.

In Model I, the power data of riders of different types and different genders is collected first. A **Fractional Model** has been established after comparison. Employing fractional model for fitting, the **power curves** and corresponding parameters of different riders are obtained.

In Model II, we first collected the course maps of 2021 Olympic and UCI World Championship, and extracted data such as curves and slopes. We also designed the track of 2022 MCM Time Trial. Based on the **Pontryagin's Maximum Principle**, a mathematical model has been established and used to deal with practical problem in sections. An Accelerating-Maintaining strategy is proposed to minimize the race time. In our practice, a female time trial specialist and a sprinter who are non world class racers can finish the Olympic ITT in **30'28"** and **31'45"**, which guarantee them meritorious performances in the Olympic feasts. By solving the optimization problem, the **Power Distribution** of the rider on each section of the course is obtained when the total energy is limited.

In Model III, we choose **Wind** and **Rainfall** to analyze the impact of weather factors. For the impact of wind, side force and air resistance are considered. For the impact of rain, we adjusted the friction coefficient in the model. The power distribution under different weather conditions is obtained from simulation and compared with the case of no wind or rain.

In Model IV, the deviation between the actual power and the ideal power is considered as a **Random Variable**. Through the random test, we get the actual power curve. On this basis, the influence of deviation on the results is analyzed.

In Model V, a Multi-leader strategy is established for TTT. Firstly, the team of six riders should form a line. Secondly, time trial specialists are recommended to be the leaders and sprinters are perfect to take the 3rd or 4th place. What's more, arranging multiple leaders to take the front lead in turns can reduce the finishing time. As a result of the simulation, the improvement by co-leader and three-leader strategy are respectively 7.4% and 9.1%. Finally, principles are summarized from the mathematical model and a proper guidance **"Cycling Tips"** is proposed for team leaders.

Keywords: Time Trial, Power Distribution, Partition Strategy, Optimization, Multi-leader Strategy

Contents

1 Introduction

1.1 Background

Road bicycle racing is the most popular professional form of bicycle racing, with a large number of competitors, events and spectators. One of the most common competition formats of road bicycle racing is time trials, where individual riders (ITT) or teams (TTT) race a course alone against the clock. Some competitions have attracted the attention of people all over the world, such as 2021 Olympic Time Trial Course in Tokyo, Japan and 2021 UCI World Championship time trial course in Flanders, Belgium.

For individual riders, they can produce different levels of power for different lengths of time. Based on different explosiveness and persistence, riders can be divided into several types, such as a **Time Trial Specialist**, a **Climber**, a **Sprinter**, a **Rouleur**, or a **Puncheur**, and each type of rider has a distinct power curve. Reasonable planning and making good use of the characteristics of each rider will help to achieve good results in the race.

1.2 Our Works

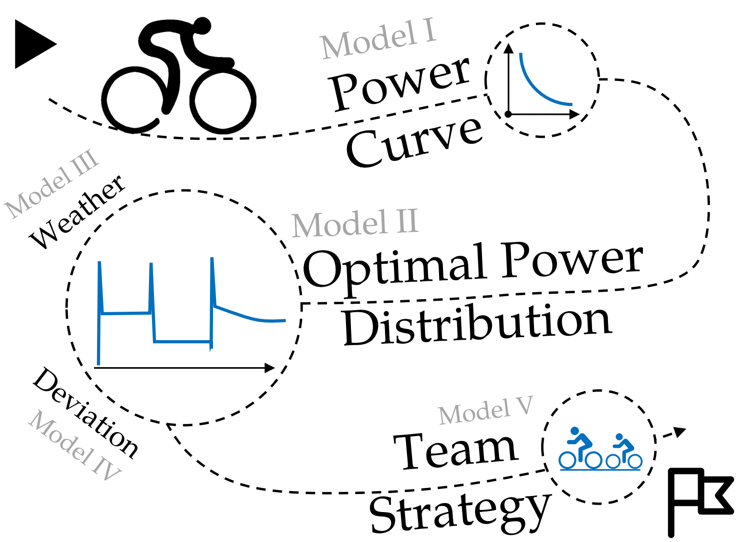


Figure 1: Overview of Mathematical Models

In this paper, to evaluate different types of riders and give the optimal strategy in ITT & TTT race, we establish mathematical models as Figure 1 shown, and complete the following works:

Work 1: Measure the **Power Curve** of riders of different types and different genders;

Work 2: Apply our model in actual courses, and obtain **Power Distribution** of minimum time;

Work 3: Consider the potential impact of **Weather Conditions** in power distribution;

Work 4: Study the **Sensitivity of Deviations** from target power distribution;
Work 5: Extend the model to include the optimal power use for a **Team Time Trial**;
Work 6: Finally, propose a rider's **Race Guidance** for a Directeur Sportif of a team.

2 Model Preparation

2.1 General Assumptions

Assumption 1: The rider meets the power curve in the race, and the power maintained will not exceed the corresponding time on the power curve;

Assumption 2: The total energy consumed by riders in a race is limited. If this energy is exceeded, the rider will no longer be able to ride;

Assumption 3: The total distance is not too short and the slopes are not too steep. This is to guarantee that it is impossible to maintain peak level for the entire trial and the critical power level suffices to achieve a positive velocity.

Assumption 4: The starting speed of the rider is a small positive value. This is to ensure that the zero point in the constraint will not appear when solving the optimization problem.

2.2 Symbol Explanation

Table 1: Symbol Explanation

Symbol	Description	Unit
v	The velocity of the rider	$\mathrm{m/s}$
m	Total mass of riders and bicycles	kg
P	The power of the rider	W
P_0	The maximum power of aerobic energy supply	W
T	Time for the rider to complete the race	s

2.3 Data Collection and Processing

2.3.1 Power of Different Types of Riders

When measuring power, four very important values is considered:

- **Sprint Abilities**, measured by the maximum power over 5 seconds;
- **Anaerobic Capacity**, measured by the maximum power over 1 minute;
- **Maximal Oxygen Consumption (VO2)**, measured by the maximum power over 5 minutes;
- **Functional Threshold Power (FTP)**, measured by the maximum power over 20 minutes.

Those four numbers divided by weight are telling us about the rider's talent in cycling. In the Article [1], the power standards of different types of male riders are obtained. However, the data of female riders is missing. To fill in the missing data, we refer to the rating table of male and female riders in the Article [1]. The data of riders are shown in Table 2.

Table 2: The Power Standards of Different Types of Male and Female Riders (W/kg)

Gender	Type	5 sec.	1 min.	5 min.	20 min.
Male	Puncheur	16.89	10.12	5.84	4.35
	Rouleur	19.85	9.55	5.74	4.71
	Sprinter	21.03	9.09	4.81	3.91
	Climber & Time Trial Specialist	16.89	8.74	5.53	4.98
Female	Puncheur	13.39	8.2	5.16	3.8
	Rouleur	15.54	7.75	5.02	4.13
	Sprinter	16.4	7.39	4.17	3.39
	Climber & Time Trial Specialist	13.39	7.11	4.87	4.38

2.3.2 Route Situation of Time Trial Courses

As a practice of our model, we apply the model to actual time trial courses: 2021 Olympic and 2021 UCI World Championship. Before that, the information related to the competition route needs to be obtained. We searched on Website [2] and got the data we wanted.

In 2021 UCI, The altitude change of route is negligible, so only the turning point needs to be considered. The distance between the turning point and the starting point is shown in Table 3.

Table 3: The Route Situation of 2021 UCI World Championship Time Trial

Gender	Data							
Male	Point	Turn 1	Turn 2	Turn 3	Turn 4	Turn 5	Turn 6	End
	Distance (km)	9.9	12.7	17.8	23.6	36.3	42.2	43.3
Female	Point	Turn 1	Turn 2	Turn 3	Turn 4	End		
	Distance (km)	9.9	12.7	17.6	21.2	30.3		

In 2021 Olympic, male riders ride one more lap than women. The measured data of turning point and slope of one lap are shown in Table 4.

Table 4: The Route Situation of 2021 Olympic Time Trial

Point	Turn 1	Turn 2	-	Turn 3	-	Turn 4	Turn 5
Slope (°)	-1.45	-1.45	0	0	0	1.87	1.87
Distance (km)	1	2.6	3.3	4.2	4.7	4.9	9.9

Point	Turn 6	Turn 7	-	Turn 8	-	End
Slope (°)	-1.29	-1.29	2.58	2.58	0	0
Distance (km)	10.4	14.8	15.4	17	17.4	22.1

Finally, we design a route for a fictional race called **2022 MCM Time Trial** as Figure 2 shows. Data has been marked in the figure.

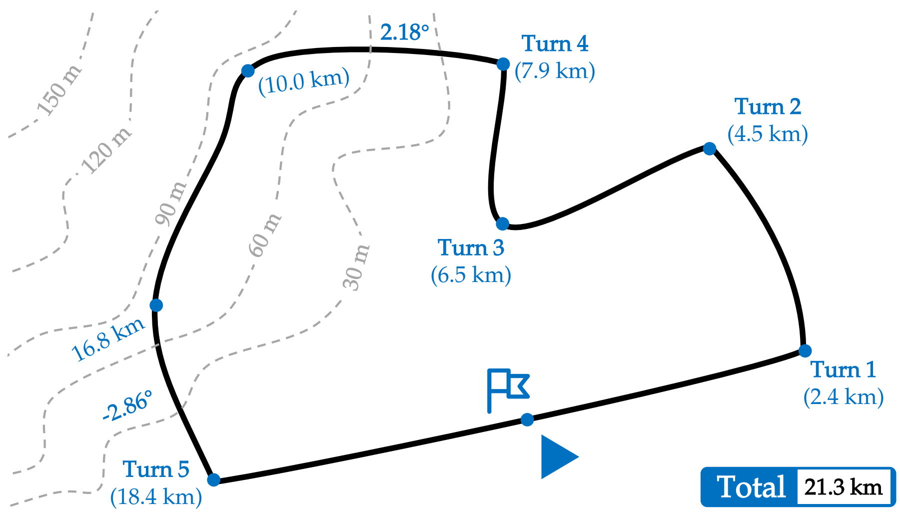


Figure 2: Designed Route for 2022 MCM Time Trial

3 Model I: Power Curve of Riders

3.1 Model Establishment and Model Selection

A rider's **Power Curve** shows how long a rider can produce a given amount of power. For a particular length of time the power curve provides the maximum power a rider can maintain for that given time. Hereinafter, we will not distinguish between power curve and power profile, and they are collectively referred to as power curve for ease of understanding. Our power curve model has the theoritical basis of energy supply systems.

According to Paper [3], in the process of movement, the energy supply process is determined by three energy supply systems, which are **ATP-PC**, **Glycolysis** and **Aerobic energy systems**. The maximum power output is equal to the sum of these three terms, i.e.

$$\text{Power Output} = \text{ATP-PC} + \text{Glycolysis} + \text{Aerobic energy systems}.$$

From Figure 3, we find that the power curve can be approximated as a downward convex curve, and the greater the power, the shorter the maintenance time.

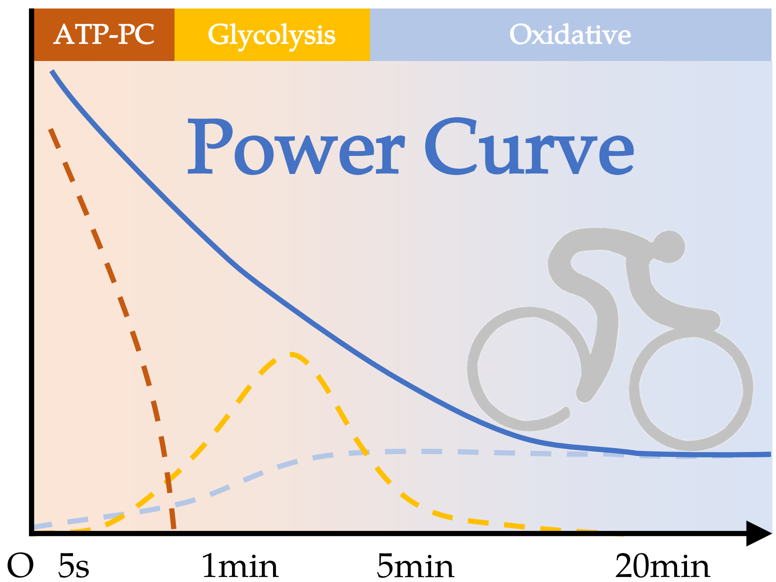
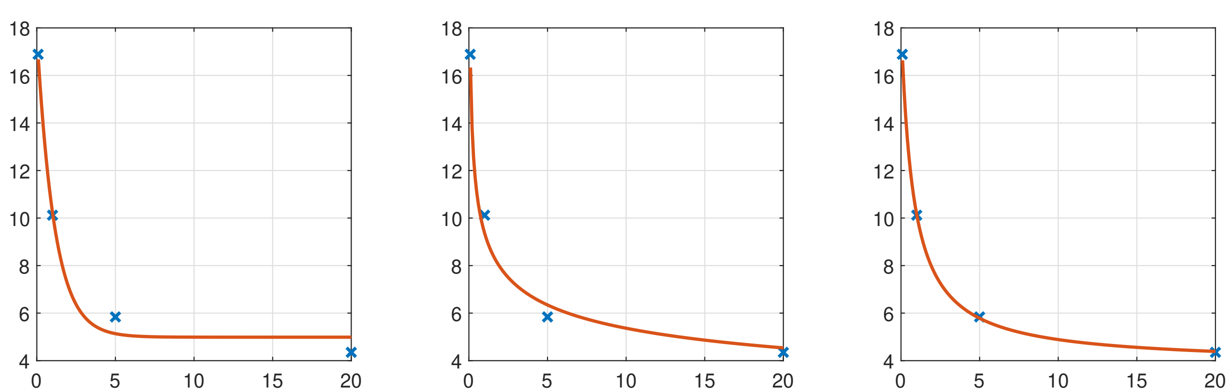


Figure 3: The Energy Supply System of Riders

Let P be power per weight (divide by the rider's weight to remove its effect), and t be time. Here, we propose three different models:

- **Exponential Model:** $P = P_0 + k \cdot \mathrm{e}^{\alpha t}$, where P_0 is the maximum power of oxidative energy supply, k is adjustment and $\alpha > 0$ is change rate;
- **Power Function Model:** $P = k \cdot t^{\alpha}$ from [4], where k is adjustment and $\alpha > 0$ is change rate;
- **Fractional Model:** $P = P_0 + \dfrac{k}{t + t_0}$, where P_0 is the maximum power of oxidative energy supply, k is slope and t_0 is adjustment of time.

In order to make **Model Selection** between the above three models, we substitute the actual data for calculation. The results are shown in the Figure 4.



(a) Exponential Model (b) Power Function Model (c) Fractional Model (Best)

Figure 4: The Fitting Result of Different Models (Data: Male Puncheur)

According to the fitting results, we find that fractal model fits the actual data best. And in the practical application, it is found that fractional model can better distinguish different kinds of riders, so the model we choose is **Fractional Model**.

3.2 Result: Power Curve from Fractional Model

The parameters (P_0, k and t_0) of fractional model are shown in Table 5. Here, P_0 is the maximum power of Aerobic energy supply, $P \to P_0$ when $t \to \infty$; k is the shape factor of the curve, and t_0 is an adjustment of time to make model more precise.

Table 5: Parameters of Fractional Model of Different Types and Different Genders

Gender	Type	P_0	k	t_0
Male	Puncheur	3.8517	11.1469	0.7719
	Rouleur	4.3832	7.1300	0.3777
	Sprinter	3.4821	7.5134	0.3448
	Climber & Time Trial Specialist	4.6089	5.6329	0.3752
Female	Puncheur	3.5179	8.4198	0.7707
	Rouleur	3.9454	5.2461	0.3692
	Sprinter	3.0965	5.8051	0.3530
	Climber & Time Trial Specialist	4.1618	3.9543	0.3451

The results are shown in Figure 5. In the following discussion, in order to distinguish different types of riders, the two research objects are **Time Trial Specialist** and **Sprinter**. The former has worse sprint capability, but FTP is better, while the latter is just the opposite.

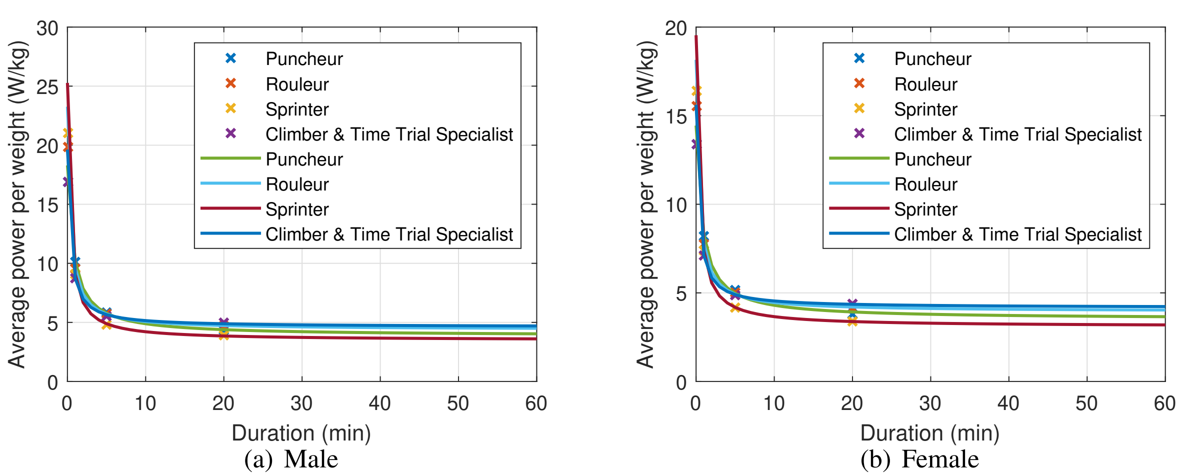


Figure 5: The Power Curve of Different Types and Different Genders

4 Model II: Optimal Power Distribution by Partition Strategy

4.1 Proposal of Optimization Problem

According to the relationship between power and speed, we have

$$P = Fv, \tag{1}$$

where v is riding speed, F is force generated by rider, and

$$F = k_A \cdot v^2 + m_e g \cdot (\sin\varphi + C_R) + m_e \cdot \frac{\mathrm{d}v}{\mathrm{d}t}. \tag{2}$$

- $k_A \cdot v^2$ is air resistance related to v, where $k_A = 0.5 \cdot C_d \cdot S \cdot \rho \approx 0.14\mathrm{N}\cdot\mathrm{s}^2/\mathrm{m}^2$. When considering the environment impact such as wind directions and wind strengths, it should be adjusted;
- $m_e g \cdot (\sin\varphi + C_R)$ is sum of rolling friction and resistance caused by gravity, where total weight $m_e = m + m_b$, m is weight of rider, m_b is weight of bike, φ is slope angle of road, and C_R is coefficient of rolling friction. When we consider the impact of rain, it should be adjusted;
- $m_e \cdot \dfrac{\mathrm{d}v}{\mathrm{d}t}$ is inertial force in non inertial frame.

Equation (1) and (2) establishes the relationship between power P and velocity v. The former is related to energy consumption and the latter is related to the distance of movement. When establishing the model of power distribution, it is limited that (a)the movement of the rider meets the power curve obtained in Model I; (b)the energy consumed by the rider during the race shall not exceed W_0; (c)at the beginning of riding, velocity v is a small value $\alpha > 0$. These ensure that there is no zero point in the constraint when $t = 0$.

The route of cycling competitions are generally composed of 2 types of shape: straight and sharp turn, where bends with radius more than 14 meters are considered to be straight for its impact on velocity reduction is proved to be tiny enough to ignore. When passing through a sharp turn with radius r, it is the static friction that provides centripetal force, and the maximum velocity in the bend is determined by:

$$\mu m g = \frac{m v_m^2}{r}.$$

where μ is static friction coefficient. Take v_m as the boundary condition for the optimization problem, so that a complete route shall be segmented into sections by adjacent sharp turns.

Let P_0 be be the power of oxidative supply, S be the total distance. The purpose of time trial is to finish the race in the shortest time, so the optimal problem we have to solve is

$$\min_P T, \quad \text{s.t.} \quad \begin{cases} P = \left(k_A \cdot v^2 + m_e g \cdot (\sin\varphi + C_R) + m_e \cdot \dfrac{\mathrm{d}v}{\mathrm{d}t}\right) \cdot v, \\ \displaystyle\int_0^T (P - P_0)\mathrm{d}t \leq W_0, \\ \displaystyle\int_0^T v\mathrm{d}t = S. \end{cases} \tag{3}$$

Here, P meet the power curve, boundary condition are $v(0) = \alpha > 0$ and $v(s_i) = \sqrt{\mu g r_i}$, where s_i and r_i represent the distance and radium of the No.i sharp turn.

4.2 Analysis of Optimization Problem

We use the solution method in [5] and [6] to solve optimization problem (3). For simplicity, let constants $c_1 = k_A$, $c_2 = mg \cdot (\sin\varphi + C_R)$ and $c_3 = m_e$. Let $u_0 = P_0$, $u \equiv P$, $x_1 \equiv x$, $x_2 \equiv v$, and let $x_3 \equiv W(t)$ represents the rest of energy. With Equation (1) and (2), we have

$$\begin{cases} \dfrac{\mathrm{d}x_1}{\mathrm{d}t} = x_2, \\ \dfrac{\mathrm{d}x_2}{\mathrm{d}t} = \dfrac{u(t)}{c_3 x_2} - \dfrac{c_1 x_2^2}{c_3} - \dfrac{c_2}{c_3}, \\ \dfrac{\mathrm{d}x_3}{\mathrm{d}t} = u(t) - u_0, \end{cases} \qquad \text{where} \quad \begin{cases} x_1(0) = L, \quad x_1(T) = L, \\ x_2(0) = \alpha, \quad x_2(s_i) = \sqrt{\mu g r_i}, \\ x_3(0) = 0, \quad x_3(T) \leq W. \end{cases}$$

Now we need to optimize $u(t)$. The corresponding Hamiltonian function is

$$H(\boldsymbol{x}, u, \boldsymbol{\lambda}) = -1 + \boldsymbol{\lambda}^T \nabla \boldsymbol{x}.$$

According to the Pontryagin maximum principle,we knew that the costate equations are

$$\begin{cases} \dfrac{\mathrm{d}\lambda_1}{\mathrm{d}t} = -\dfrac{\partial H}{\partial x_1} = 0, \\ \dfrac{\mathrm{d}\lambda_2}{\mathrm{d}t} = -\dfrac{\partial H}{\partial x_2} = -\left[\lambda_1 - \dfrac{\lambda_2(t) u(t)}{c_3 x_2^2(t)} - \dfrac{2c_1}{c_3}\lambda_2(t) x_2(t)\right], \\ \dfrac{\mathrm{d}x_3}{\mathrm{d}t} = -\dfrac{\partial H}{\partial x_3} = 0, \end{cases}$$

and the boundary condition is $\lambda_2(t) = 0$. What we need is u^* when $H(x^*, u^*, \lambda^*)$ reaches the maximum. Note that H is a linear function of u, so the coefficient $\dfrac{\lambda_2(t)}{c_3 x_2(t)} + \lambda_3(t)$ determinds the value of u^*.

- $\dfrac{\lambda_2(t)}{c_3 x_2(t)} + \lambda_3(t) < 0$, then $u^* = u_0$ is the power of oxidative supply;
- $\dfrac{\lambda_2(t)}{c_3 x_2(t)} + \lambda_3(t) > 0$, then $u^* = u_m$ is the maximum power of anaerobic supply;
- $\dfrac{\lambda_2(t)}{c_3 x_2(t)} + \lambda_3(t) = 0$, then the corresponding u^* is power that can sustain. Let $\gamma = -c_3\lambda_3$, then

$$v^* = x_2 = \sqrt{\frac{c_3}{3c_1\gamma} - \frac{c_2}{3c_1}}, \quad u^* = \frac{(c_3 + 2c_2\gamma)}{3\sqrt{3}\gamma} \cdot \sqrt{\frac{c_3 - c_2\gamma}{c_1\gamma}}.$$

This analysis provides an optimization method of **Accelerating - Maintaining** to shrink the use of time in ITT, which reminds riders that it's necessary to perform rapid acceleration at first. When a rider's velocity rise to v^*, his power output should be lowered down to u^*, which guarantees they can make full use of the limited W_0. Contradicting to our commonsense, if pursuing the minimum time, the limited energy should be used up and there is no need of remaining to support the last sprint. The truth is that it might only works psychologically that racers often choose to sprint at the end of races.

4.3 Result: Power Distribution in Real Races

The predicted finishing time is shown in Table 6.

Table 6: Finishing Time of Riders in Simulation

Rider	Male		Female	
	TT Specialist	**Sprinter**	**TT Specialist**	**Sprinter**
2021 Olympic	58’07”	61’02”	30’28”	31’45”
2021 UCI World Championship	52’46”	56’47”	44’21”	48’03”
2022 MCM	56’57”	60’40”	29’49”	31’26”

4.3.1 2021 Olympic Time Trial course in Tokyo, Japan

For 2021 Olympic, the data of male an female time trial specialists and sprinters are documented, the optimal power distribution strategies are shown in Figure 6. The competition route in Tokyo fluctuates greatly and there are many curves, so the power fluctuates significantly too. By distribute their energy as it is shown in figure. They can finish their races respectively in 58’07”, 61’02”, 30’28” and 31’45”. We further explore the differences between the two riders:

- Sprinters are more aggressive during the accelerating period while they lack the ability to sustain a long term of **anaerobic energy**;
- Although time trial specialists appear not strong in the ability to burst instantly, they get better results due to the higher **lactic acid tolerance**.

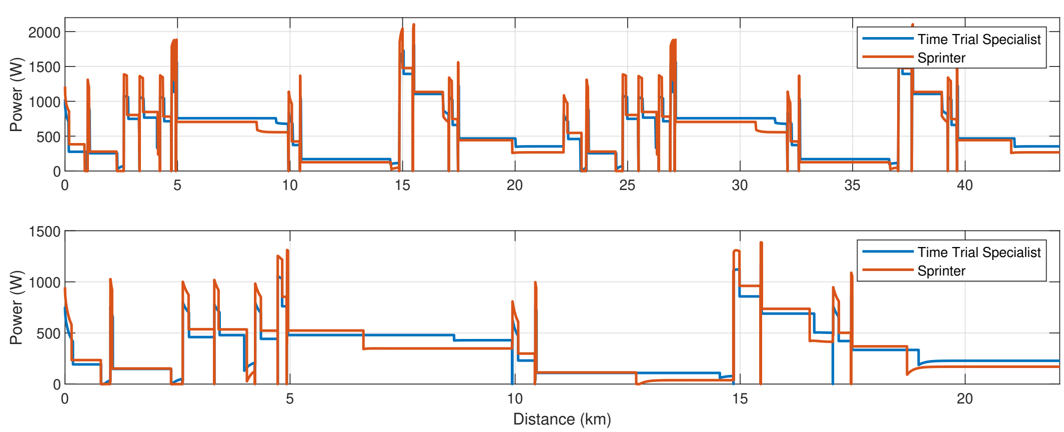


Figure 6: The Optimization Power Distribution in 2021 Olympic (Male and Female)

4.3.2 2021 UCI World Championship time trial course in Flanders, Belgium

For 2021 UCI, the optimal power distribution strategies are shown in Figure 7. There is almost no altitude change in the race route of Flanders, so the power output is more stable. Analyzing the data of the same athletes in 2021 Olympics, it is optimized that they can finish their races respectively in 52'46", 56'47", 44'21" and 48'03".

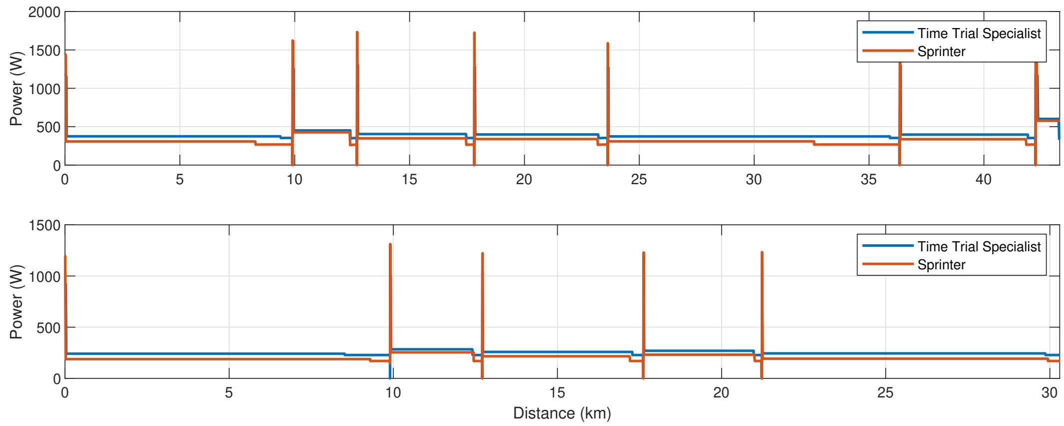


Figure 7: The Optimization Power Distribution in 2021 UCI (Male and Female)

4.3.3 Designed Route for 2022 MCM ITT

For the designed route of 2022 MCM ITT (Figure 2), the optimal power distribution strategies are shown in Figure 8. Still employing the data of the same athletes in 2021 Olympics, it is optimized that they can finish their races respectively in 56'57", 60'40", 29'49" and 31'26".

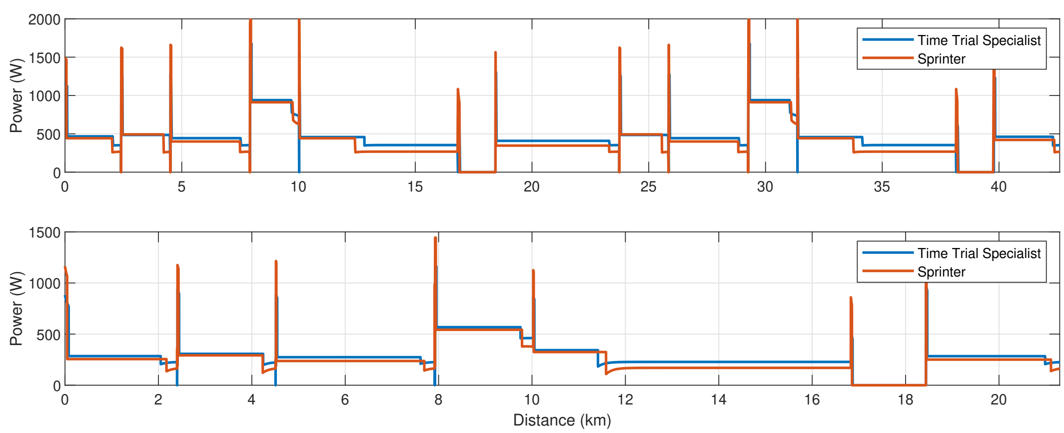
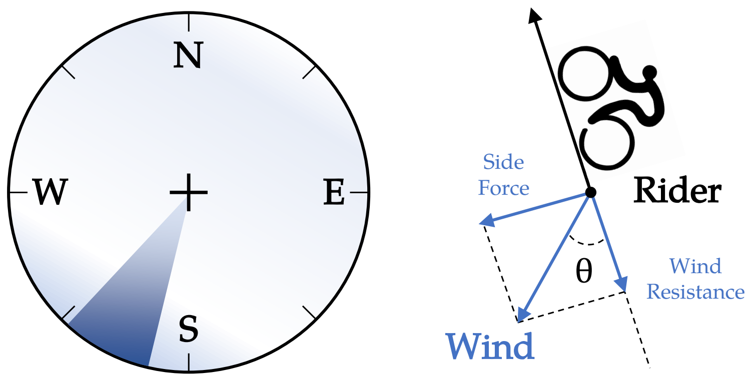


Figure 8: The Optimization Power Distribution in 2021 MCM ITT (Male and Female)

5 Model III: The Impact of the Environment

5.1 Wind Directions and Wind Strengths

To analyze the influence of wind and wind direction on riders, Equation (2) should be modified.



(a) Wind Graph (b) Force Analysis of Rider

Figure 9: Analysis with Wind Directions and Wind Strengths

Under crosswind conditions, the wind speed causes two components of force: wind resistance F_A in the direction of the relative speed between the rider and the wind and side force F_S in the direction perpendicular to the rider's speed (see Paper [7]). Here, it is defined that

$$k_A = 0.5 \cdot C_d \cdot A \cdot \rho, \quad \text{and} \quad k_S = 0.5 \cdot C_S \cdot A \cdot \rho,$$

where C_d and C_S are the drag force coefficient and the side force coefficient (both from experimental results), A is the total frontal area of the cyclist and the bicycle at yaw angle and ρ is the air density at $25°\text{C}$. Then, F_A and F_S can be expressed as

$$F_A = k_A \cdot v_w^2, \quad \text{and} \quad F_S = k_S \cdot v_w^2.$$

Table 7: The Value of k_A and k_S when Angle Changes (from [7])

Angle (°)	**0**	**15**	**30**	**45**	**60**	**75**	**90**
k_A	0.3	0.315	0.283	0.21	0.175	0.098	0.01
k_S	0	0.15	0.364	0.553	0.644	0.721	0.75

- **The Impact of Drag Resistance**

  The air resistance F_A is now $k_A \cdot (v + v_m \cos\theta)^2$, so in the tangential direction of the rider's velocity v, we have

$$F = k_A \cdot (v + v_w \cos\theta)^2 + m_e g \cdot (\sin\varphi + C_R) + m_e \cdot \frac{\mathrm{d}v}{\mathrm{d}t}, \tag{4}$$

  where v_w is the speed of wind, and θ is the angle between the wind and the opposite direction of motion be (as Figure 9 shown). Using Equation (4) instead of Equation (2) and solving (3) again, we can get the result we want.

- **The Impact of Side Force**

  In addition to considering the influence of wind on the direction of motion, it is also necessary to consider the lateral influence of wind. The force of the wind on the side is

$$F_S = k_S \cdot v_w^2,$$

where k_S varies when θ changes. When $\theta = 90°$, F_s reaches the maximun $F_{S,\max} = k_{S,\max} \cdot v_w^2$. When passing through sharp turns, it is the resultant force of F_s and f that provides centripetal force, i.e.

$$f - F_S = \frac{mv^2}{r}.$$

If $F_{S,\max} > \mu mg$, it's too dangerous to continue the race because riders will lose their balance at sharp turns. If $F_{S,\max} < \mu mg$, we obtain

$$v_m' = \sqrt{\left(\mu g - \frac{k_{S,\max} \cdot v_w^2}{m}\right) \cdot r},$$

which is the maximum velocity when the rider passes sharp turns.

5.2 Rainfall and Air Humidity

According to [8], humidity and rainfall have a certain impact on the friction of the ground.

Table 8: Mean BPNs of Pavement under Different Conditions (from [8])

Road Type	Dry	Damp	Moist
AC-16 Old	76.4	69.0(90%)	64.8(84%)
AC-16 New	92.6	74.8(81%)	74.0(80%)
SAM-16	95.8	80.4(84%)	74.0(77%)
OGFC-13	100.8	85.0(84%)	84.8(84%)

From Table 8, moisture will reduce the friction factor of the pavement by about 80%. In practical application, we can modify C_R of Equation (2) or (4) to measure different levels of humidity.

5.3 Result: The Impact of Environment on Riding

5.3.1 The Impact of Wind

This model takes 2022 MCM ITT (map shown in Figure 3) as an example. Considering the influence of different wind velocity and a given initial wind direction of 15°,the rider's power distribution is shown as Figure 10.

Wind strength: It is obvious from the results that when the wind is gentle($2.5\mathrm{m/s}$), the modified velocity and power output highly coincides with the results of no wind . However, When the wind is

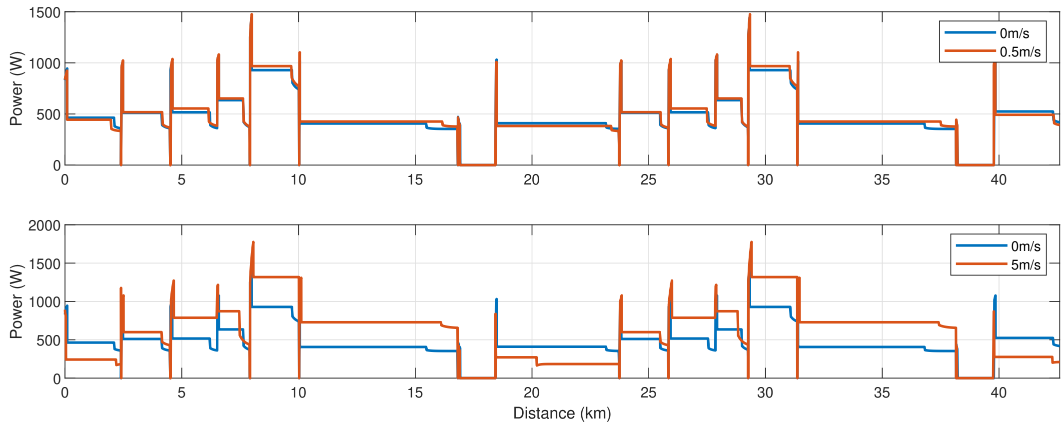
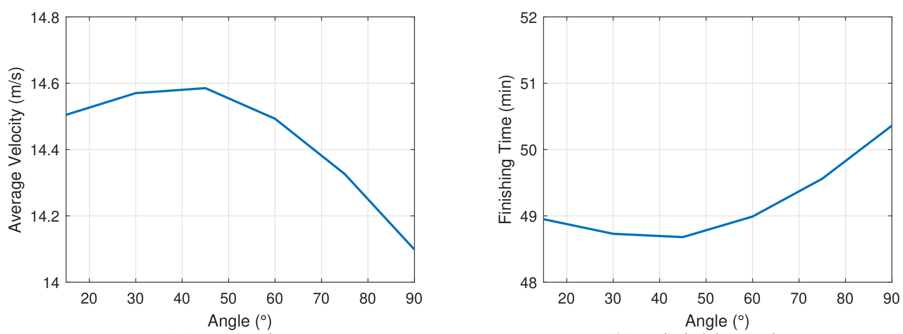


Figure 10: The Impact of Wind in 2022 MCM ITT ($2.5\mathrm{m/s}$ and $5\mathrm{m/s}$)

too strong($5\mathrm{m/s}$), the power is significantly reduced. In conclusion, this model is not sensitive to small disturbances within a reasonable range .



(a) Velocity (b) Finishing Time

Figure 11: Change of Average Velocity and Finishing Time when Wind Direction Changes ($5\mathrm{m/s}$)

Wind direction: Results are shown in Figure 11. With the change of wind direction, the average velocity and time will change to a certain extent. This change is not large, indicating that our model is proved to be not too sensitive to the impact of the environment.

5.3.2 The Impact of Rain

According to our model, the friction of the ground will be reduced during rainfall. The simulation results are shown in Figure 12. When the ground is wet, the traction decreases due to the reduced friction, which leads to the rider's faster speed. In order to maintain normal speed, riders need to consume higher power. The result is consistent with the actual situation.

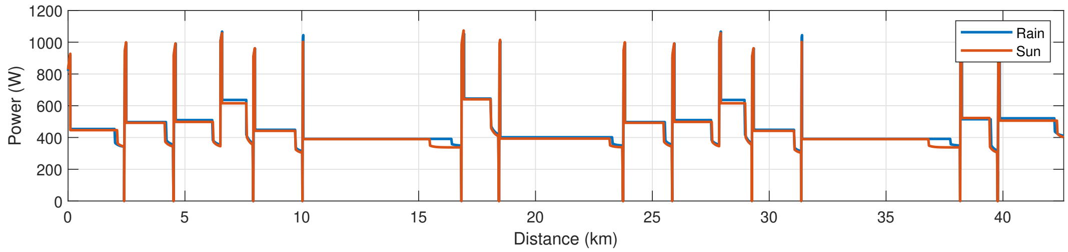


Figure 12: The Impact of Rain in 2022 MCM ITT

6 Model IV: Sensitivity of Deviations from Target Power

In practice, the target power that riders can achieve in each race has a certain randomness. Optimal power distribution can be obtained by Model II, but it is possible that riders fail to follow the power distribution in the race perfectly. It's more practical to offer the rider and the Directeur Sportif with ideas of the possible range of expected split times at key parts of a given course. The principles are:

- Remember the velocity distribution during each straight sections and at each turns;
- Never exceed the velocity limits at sharp turns to avoid losing balance;
- Accelerate at a high power immediately after leaving a turn, and try to maintain the planned velocity;
- Once the planned power output is exceeded, extra time will be spent on recovering at the power output level supplied by aerobic energy only.

6.1 Deviation Model from Target Power

We assume that the actual power P' generated by the rider is equal to the ideal power P plus a random error e ([9] and [10]), i.e.

$$P' = P + e, \quad \text{where} \quad \mathbb{E}(e) = 0 \quad \text{and} \quad \mathrm{Var}(e) = \sigma^2.$$

Here, $\mathbb{E}(e) = 0$ represents that the expectation of error is 0. This is because the rider is trained to maintain near the target power.

In order to analyze the impact on the results caused by the deviation from the target, we consider the average power $\overline{P}$ here, and let $\overline{P'}$ be the actual power. Suppose there are n errors in the riding process, and the value of each error is $e_i (1 \le i \le n)$, with the same assumption that $\mathbb{E}(e) = 0$ and $\mathrm{Var}(e) = \sigma^2$. Then

$$\overline{P'} = \overline{P} + e, \quad \text{where} \quad e = \sum_{i=1}^{n} e_i.$$

When $0 < \sigma < +\infty$, the central limit theorem indicates

$$\frac{e_1 + e_2 + \cdots + e_n - n \cdot 0}{\sqrt{n} \cdot \sigma} \xrightarrow{d} \mathcal{N}(0, 1),$$

hence e approximately obeys normal distribution $\mathcal{N}(0, n\sigma^2)$. Suppose the total energy is W_0. We know that power is consumption rate of energy, so we can choose time to analyze the impact of deviation from the target power on the results. The actual riding time and its ratio to ideal time are

$$t' = \frac{W_0}{\overline{P'}} = \frac{W_0}{\bar{P} + e} \quad \text{and} \quad \frac{t'}{t} = \frac{\bar{P}}{\bar{P} + e}.$$

6.2 Result: Simulation of Deviation and Influence of Deviation

It is common sense that errors are usually subject to normal distribution. Let $e \sim \mathcal{N}(0, \sigma^2)$ i.e. normal distribution with mean value of 0 and variance of σ^2. Add the error term to the given power distribution, and the results are shown in Figure 13. From Figure 13, we can see that the actual power will revolve around the ideal power, but there will be some fluctuations.

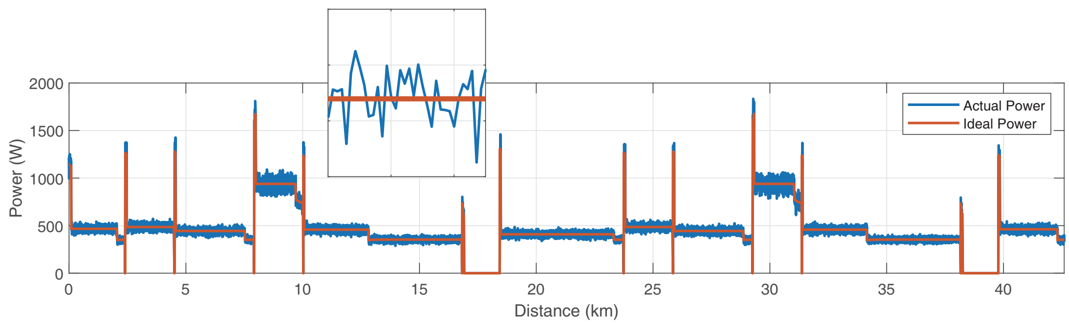
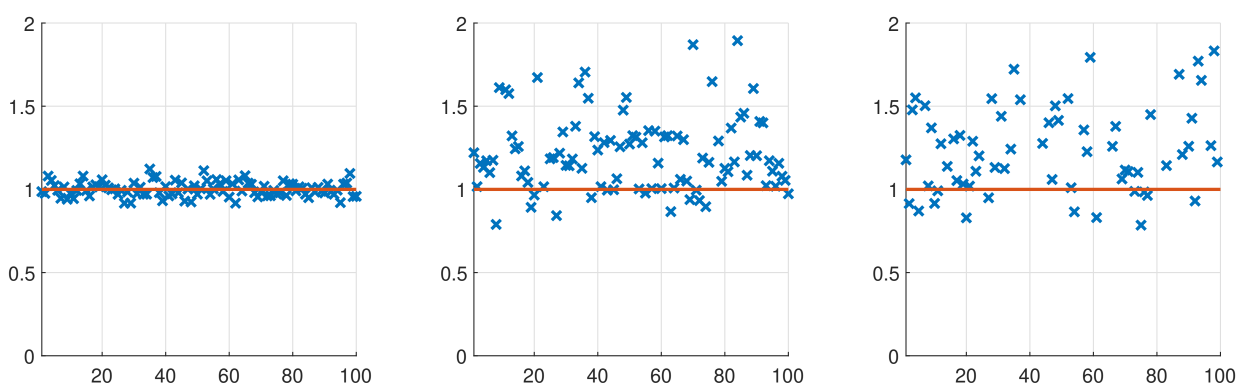


Figure 13: Comparison between Actual Power and Ideal Power ($\sigma = 0.05$)

We further consider the influence of deviation on the results of the model. Ratio of actual time to ideal time when $n = 1, 5, 20$ are shown in Figure 14.



(a) $n = 1$ (b) $n = 5$ (c) $n = 20$

Figure 14: The Deviation of Results ($\sigma = 0.05$)

- In most experiments, the actual time is close to the ideal time;

- If the number of mistakes n is large, the deviation of the actual time from the ideal time will be greater, because the variance $n\sigma^2$ is larger;
- If there is a certain deviation from the ideal power, the time is more likely to be long, which makes the performance of the game worse.

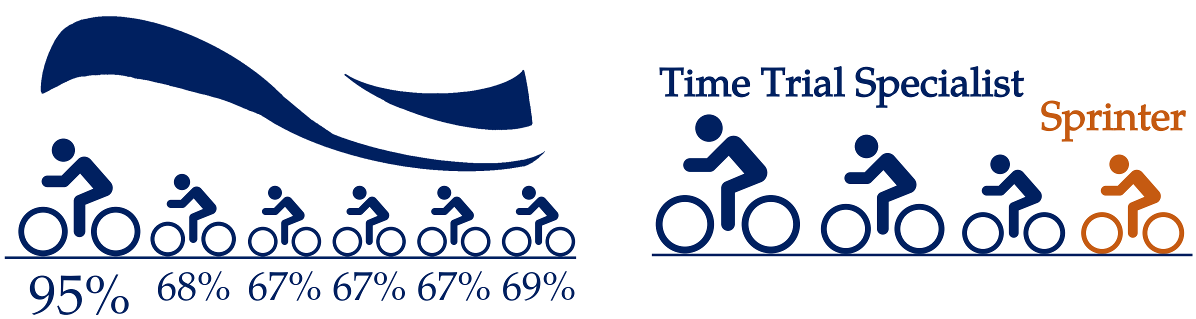
7 Model V: Best Strategy for Team Time Trial (TTT)

7.1 Team Working Methods of Multi-leader strategy

Here, we established a series of team working methods in order to extend our model to include the optimal power use for a team time trial of six riders per team. The team's time is determined when the fourth rider crosses the finish line.

7.1.1 Method 1: Form a Line

Research [11] shows that riding in a line reduces the air resistance to certain extents for different positions in the team. As for a six-member line, the reduction of air resistance compared to individual riding is shown as Figure 15. Thus, it's concluded that in a team time trial, the air resistance against the leader and the forth place in the team have been reduced to 95% and 67%, separately.



(a) Air Resistance against Riders (b) Position Distribution under Tri-leader Strategy

Figure 15: Strategy for TTT

7.1.2 Method 2: Proper Position of Each Member

As the time of the team is determined by the first four rider's finishing time, the recommended strategy is **Sprinters** (SP) following **Time Trial Specialists** (TT). The velocity of the team is determined by the leader who is subject to a 95% air resistance compared to ITT, the highest among the team. Meanwhile, the energy of the 2nd to the 4th riders is greatly saved to make the last sprint. As is analyzed in the establishment of Power Curves, sprinters are proved to have excellent ability to sprint while they lack the capacity to sustain a long term of anaerobic energy, which makes them perfect for the 3rd or 4th place. With leaders ahead reducing the air resistance for them, sprinters will achieve a faster finishing time by doing the last sprint using the saved energy.

7.1.3 Method 3: Multiple Leaders in turn

Based on the position distribution of members, we proposed the Multi-leader strategy to further improve the finishing time. Single-leader strategy which have only one leader in the front to combat the air resistance will drive the leader too exhausted to carry out the last sprint. However, a co-leader or tri-leader strategy will largely reduce the average air resistance against each leader by taking turns leading in the front. It is concluded that the **Tri-leader Strategy** is the optimal according to the following analysis

7.2 Result: Application of the Strategy

In a cycling line formed by six members, the air resistance against each member reduces to different extent, thus

$$F_A = \beta_i \cdot k_A \cdot v^2,$$

where β_i, Pecentage of F_A, is shown in Figure 15. The team's velocity is determined by the front leader who bears the biggest air resistance. The modification is applied to our model with the 2022 MCM route, and the relationship between velocity and time is obtained.

Table 9 shows the results of the team's finishing time employing our Model modified by single-leader, co-leader and tri-leader strategies, respectively (the data in the brackets indicates the increase of the corresponding items compared to single-leader strategy). The data proves that by employing the muti-leader strategy, all of the first four riders can save enough energy to promote their speed when sprinting at the last section of the course and tri-leader strategy achieves the best finishing time.

Table 9: The Comparison between Muti-Leader Strategies

Strategy	Single-leader	Co-leader	Tri-leader
β_i	95%	81.5%	76.7%
Sprinting speed $(\mathrm{m/s})$	14.97	15.75($+5.2\%$)	16.07($+7.3\%$)
Finishing time	57'21"	53'10"(-7.4%)	52'14"(-9.1%)

8 Conclusion

8.1 Summary of Models and Results

In summary, we established mathematical models as Figure 1 shown. Based on our models, we evaluate different types of riders, analyze the influence of weather and deviation, and give the optimal strategy in ITT and TTT competitions.

- **Model I: Power Curve of Riders**

  Based on the data of Table 2, we use exponential model, power function model and fractional model to fit. After comparison (see Figure 4), we find that the fractional model has the best fitting effect and can more effectively distinguish different types of riders. Therefore, the model we choose is the fractional model, i.e.

$$P = P_0 + \frac{k}{t + t_0},$$

  where P_0 is the maximum power of oxidative energy supply, k is slope and t_0 is adjustment of time. Parameters and results of the model are shown in Table 5 and Figure 5.

- **Model II: Optimal Power Distribution by Partition Strategy**

  We collected the data of 2021 Olympic and 2021 UCI track (as shown in Table 3 and 4), and designed the map of 2022 MCM ITT (as shown in Figure 2). Based on the power curve, we propose an optimization problem under the condition that the energy consumed by the rider is limited and meets the power curve:

$$\min_{P} T, \quad \text{s.t.} \quad \begin{cases} P = \left(k_A \cdot v^2 + m_e g \cdot (\sin\varphi + C_R) + m_e \cdot \dfrac{\mathrm{d}v}{\mathrm{d}t} \right) \cdot v, \\ \displaystyle\int_0^T (P - P_0)\mathrm{d}t \le W_0, \\ \displaystyle\int_0^T v\mathrm{d}t = S. \end{cases}$$

  Solving by segments, we get the theoretical finishing time (see Table 6) and ideal power distribution (see Figure 6, 7 and 8) and the time to complete the game of riders. The results show that time trial specialist has better competition performance.

- **Model III: The Impact of the Environment**

  Further, we analyze the impact of the environment on riders. On the one hand, in order to consider the influence of wind, we divide the wind into lateral (F_S) and moving directions (F_A). Considering the wind speed, we revised the above model and recalculated it; On the other hand, in order to discuss the impact of rain, we consult the literature to obtain data in Table 8, and reduce the friction factor to 80%. Results of the model are shown in the Figure 10 and 12.

- **Model IV: Sensitivity of Deviations from Target Power**

  To analyze the sensitivity of the model, we assume that the error between the actual power distribution and the ideal power distribution is a random variable. First of all, we simulated the deviation from the target, as shown in Figure 13. On the basis of deviation, we analyzed the change of competition time and found that the more deviation times, the worse the performance of the competition see Figure 14.

- **Model V: Best Strategy for Team Time Trial (TTT)**
  We established a series of team working methods in order to extend our model to include the optimal power use for a team time trial (TTT). Here, we propose three strategies: Form a line, find proper positions for each member and use multi-leader strategy. Our strategy can effectively improve the performance of the game. For details, see Table 9.

8.2 Model Strengths and Weaknesses

8.2.1 Strengths

The model we built has the following advantages:

- **Theory-supported power curve model.** We established a power curve model based on the theory of energy supply systems that is highly consistent with the data we obtained.
- **The model is reasonably simplified and has a strong universality.** Our model shows strong stability and simplicity for extension in all the conditions of wind strength and direction, rainfall, individual deviations and team races.
- **The results of the model are in line with reality with considerable advantages.** Our simulation results ranked among the top in the application of international races, which shows the effectiveness of our strategy.
- **The strategy we proposed for Team Time Trial is creative and imaginative.** We designed a Tri-Leader strategy to save more energy of both the leaders and the sprinters for the last sprint and it is proved that this strategy can reduce the finishing time by 9.1%.
- **The model has advantageous guiding significance to the riders and the Directeur Sportif.** The competition principles of ITT and TTT can be summarized from the mathematical model, which plays a guiding role in the training of riders.

8.2.2 Weaknesses

However, our models also have some defects and remain to be improved.

- Due to the computational force, the accuracy of the model is limited. $\mathrm{d}t$ in the model can be further reduced to make the power allocation more accurate.
- In the case of TTT, the power curves of six athletes were not optimized due to the limited computing power. In the next model, modelers can consider using multivariable optimization method or change a better computer to solve this problem.

CYCLING TIPS FOR TEAM LEADERS

The purpose of this guidance is to help Directeur Sportif and riders prepare for the 2022 MCM Team Time Trial (TTT) Competition, particularly for leaders in the team. The theoretical support of this article comes from our paper, written for 2022 Mathematical Competition in Modeling.

The **2022 MCM TTT** is a competition for teams made up of 6 riders, with time recording by the forth rider to cross the finishing line. We have recommended sprinters to take the third or forth place to and time trial specialists to play leaders. The leader undertakes crucial tasks to manage the direction and velocity as well as bearing the wind resistance for the whole team. We will show you in detail how to achieve considerable improvement based on the results and proposed strategy of our model in the following part of this guidance.

1. What is Power Curve and Power Distribution?

Power Curve tells us about your capacities in cycling. It indicates how long a rider can ride under a given power. Generally, some of the riders are more explosive and sprint faster (e.g. sprinters), while some riders perform better in endurance and physical strength (e.g. time trial specialists).

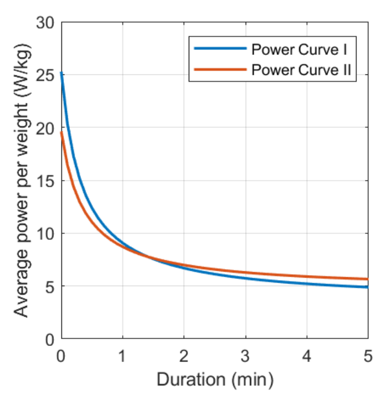


Figure 1: Two Types of Power Curves

Based on power curves, we established an optimal **Power Distribution** model with force & power analysis and mathematical optimization, which clearly shows the relationship between your position on the course and the optimal power to apply.

We've recommended *time trial specialists* to be the leader for their better control of the velocity and power output. Besides, with them holding the air resistance for the whole team, the sprinter can conserve his energy to a maximum for the last sprint.

2. What Velocity Distribution Strategy Should Leaders Employ?

The MCM Time Trial Course consists of 2 types of shape: straight and sharp turn (bends with radius more than 14 meters are considered to be straight, for the strategy we recommend would be of no difference). In our model, the course is segmented

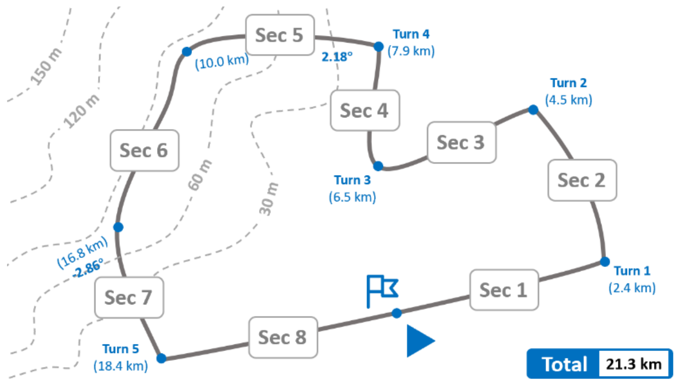


Figure 2: The Course of 2022 MCM TTT

into 8 sections with every two adjacent sharp turns, as is shown in *Figure 2*. Each section employs the similar strategy of **Accelerating – Maintaining**.

The velocity of your team is largely determined by the control of the leader. In an optimal pacing strategy, your team need to go all out at the beginning until you reach a velocity that can be maintained for the entire section. Then, it's practical for the team to lower your power output only to keep the planned velocity. As the velocity is limited for turning around a sharp turn, your team should better lower your power output from a distance to meet the velocity limit. The optimal power distribution for each section with proper velocity is shown as *Figure 3*.

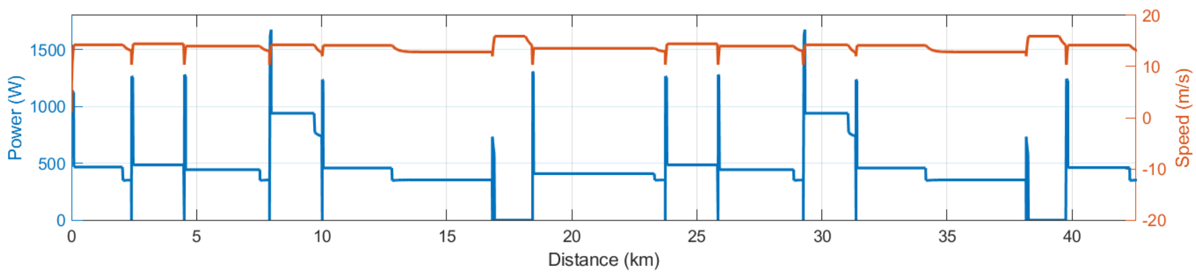


Figure 3: The Optimal Power Distribution and Velocity Distribution for Each Section

3. How to Promote the Velocity with Team Work?

Riding in a line with leaders changing regularly. Researches indicates that riding in line will reduce the air resistance against the front leader by 5%. If there are more than one leader in your team, changing leaders regularly will better conserve their energy to speed up to a higher velocity for the last sprint. Compared to only one leader in a team, co-leader strategy and tri-leader strategy will reduce the finishing time by 7.4% and 9.1%. It's recommended to perform the change of leader during the 'Maintaining' part of a section, which limits the resume of energy to a minimum in such processes.

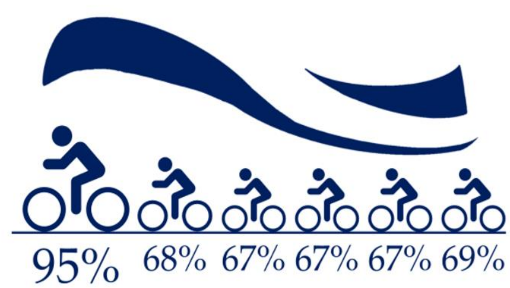


Figure 4: The Impact of Wind on Each Rider

Wish you achieve outstanding results in the 2022 MCM Time Trial!

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Appendices

A Code: Power Curve of Riders

```
t = [1/12 1 5 20];
tspan = 0 : 60;
p = csvread('Curve.csv', 1, 2);
fun = @(k, t) k(1) + k(2) ./ (t + k(3));
for i = 1 : 8
    k(i, :) = lsqcurvefit(fun, [4 500 30], t, p(i, :));
    grid on;
end
```

B Code: Optimal Power Distribution

```
function [p_acc0, t_acc0, vtq0, t_total0] = velocity(p0, k0, t0, CP, W, N,
   v_in, L, cos, v_w, alpha, slope, c1, t_total, c3, dt, dp, vtq)
c2 = 3.750 + 83.9 * sin(slope/180*pi) - 83.9 * 0.005 * alpha;
for t_acc = 10 : dt : 200
    for p_acc = p0 + k0/((t_acc + 5) / 60 + t0) : dp : p0 + k0/((t_acc + 2)
       /60 + t0)
```

```
        vtq(1) = v_in;
        for i = 1 : (floor(t_acc/dt))
            vtq(i+1) = (-(c2/c3) + p_acc*75/(c3*vtq(i)) - (c1*(vtq(i) + cos*
               v_w)^2)/c3)*dt + vtq(i);
        end
        p_sin = (c1*(vtq(floor(t_acc/dt)) + cos*v_w)^2 + c2) * vtq(floor(
           t_acc/dt));
        t_plate = (W - t_acc*(p_acc*75 - CP))/(p_sin - CP);
        if t_plate <= 0
            vtq(:) = 0;
            break;
        end
        for i = floor(t_acc/dt) : floor(t_plate/dt) + (floor(t_acc/dt))
            if (sum(vtq) - vtq(i+1))*dt >= L
                count = i;
            if count*dt < t_total
                t_total0 = count*dt;
                count0 = count;
                t_total = count*dt;
                t_acc0 = t_acc;
                p_acc0 = p_acc;
            end
            vtq(:) = 0;
            break;
        end
        vtq(i+1) = vtq(i);
        end
        if sum(vtq) == 0
            continue;
        end
    end
end
```